On the quasi-static theory of space-time fractional thermoelasticity in the unbounded domains

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Abstract

The thermal conduction phenomenon is known to be strongly dependent on the temperature and the inner structure of materials. In some materials that include impurities and/or voids, thermal conductivity deviates from the conventional behaviour. In this work, anomalous thermal conductivity that deviates from the classical definition is mathematically expressed through a space-time fractional Fourier law of heat conduction. A quasi-static theory of fractional thermoelasticity that uses a fractional space-time fractional Fourier law is considered in this work. An initial value problem is solved on the unbounded domain with initial conditions on the temperature and stresses concentrated on the middle plane that separates the two half-spaces constituting the full unbounded space. It is found that such initial conditions need an additional boundary condition in the infinity so that the initial value problem transforms into an initial boundary value problem. Exact solutions for the temperature and displacement are derived in terms of the Fox H-function. Graphical representations for the temperature and displacement show that the anomalous thermal conductivity proposed in this work has a significant effect on both the temperature and displacement distribution.

Keywords: Thermoelasticity, Fractional derivative, Quazi-static, Closed form solution, Mittag-Leffler function, H-function.

النظرية شبه الساكنة للمرونة الحرارية ذات الكسور الزمنية والمكانية في المجالات غير المحدودة

المستخلص

من المعروف أن ظاهرة التوصيل الحراري تعتمد بشكل كبير على درجة الحرارة والهيكل الداخلي للمواد. في بعض المواد التي تحتوي على الشوائب والمسام، ينحرف التوصيل الحراري عن السلوك التقليدي. في هذا العمل، يتم التعبير رياضيا عن التوصيل الحراري الشاذ الذي ينحرف عن التعريف الكلاسيكي من خلال قانون فوريير الكسري الزماني والمكاني للتوصيل الحراري. تم النظر في نظرية شبه ثابتة للديناميكا الحرارية الكسرية تستخدم قانون فورييه الكسري الزماني والمكاني. تم حل مشكلة القيمة الابتدائية في المجال غير المحدود مع الشروط الابتدائية على درجة الحرارة والإجهادات المركزة على المستوى الأوسط الذي يفصل بين نصف الفضائيين اللذين يشكلان الفضاء غير المحدود الكامل. وجد أن مثل هذه الشروط الأولية تحتاج إلى شرط حدودي إضافي في اللانهاية بحيث تتحول مشكلة القيمة الابتدائية إلى مشكلة القيمة الابتدائية والحدودية. تم اشتقاق حلول دقيقة لدرجة الحرارة والإزاحة من حيث دالة فوكس. تُظهر التمثيلات الرسومية لدرجة الحرارة والإزاحة أن التوصيل الحراري الشاذ المقترح في هذا العمل له تأثير كبير على توزيع كل من درجة الحرارة والإزاحة.

الكلمات المفتاحية : الديناميكا الحرارية المرنة – المشتقة الكسرية – شبه ساكن – حل مغلق الشكل – دالة ميتاغ ليفلر – دالة فوكس H

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Introduction

Recent years have witnessed significant advancements in the development of a comprehensive theory of thermoelasticity. However, the traditional uncoupled theory of thermoelasticity presents two fundamental shortcomings that are not aligned with observed physical phenomena. Firstly, a significant limitation of the classical uncoupled theory of thermoelasticity lies in its heat conduction equation, which lacks any elastic terms. This omission implies that thermal and mechanical phenomena are entirely independent, such a simplification does not accurately reflect the complex interdependence between temperature changes and deformations in real materials. Secondly, the parabolic nature of the heat equation within this theory leads to a counterintuitive prediction: infinite speeds of propagation for heat waves. This contradicts experimental observations, which demonstrate that heat transfer occurs at finite speeds. To address these shortcomings, Duhamel [1] pioneered the consideration of elastic problems in conjunction with heat changes, laying the groundwork for more comprehensive and realistic models of thermoelastic behaviour. Biot [2] proposed the coupled theory of thermoelasticity to rectify the first shortcoming of the classical uncoupled theory. This theory establishes a connection between the equations of elasticity and heat conduction, thereby addressing the issue of the independent treatment of thermal and mechanical phenomena. However, both the coupled and uncoupled theories share a common limitation: the parabolic nature of their heat equations, which implies infinite propagation speeds for heat waves.

To get over the first drawback, Biot [2] proposed the coupled theory of thermoelasticity. Since this theory's elasticity and heat conduction equations are connected, the first paradox of the traditional uncoupled theory. But the second flaw in both hypotheses is the same. since the linked theory's heat equation is likewise parabolic. Cattaneo [3] was a pioneer in addressing the fundamental flaw of Fourier's heat conduction law, which incorrectly predicts infinite propagation speeds for thermal disturbances. By modifying the classical law, Cattaneo's theory paved the way for the concept of thermal waves, which propagate at finite speeds, aligning more closely with observed physical phenomena.

Building upon Maxwell's earlier insights [4] and Cattaneo's groundbreaking work, a substantial body of research [5], [6] has emerged, dedicated to resolving the paradox of instantaneous thermal disturbance propagation. A key approach is extended irreversible thermodynamics, which involves incorporating time derivatives of the heat flux vector, Cauchy stress tensor, and its trace into the classical Fourier law while adhering to the entropy principle. This innovative framework offers a more accurate and comprehensive description of heat conduction processes. Puri and Kythe [7] examined the impact of the Maxwell-Cattaneo model on Stokes' second problem, which involves a viscous fluid.

Several extensions of the coupled theory have been proposed. Hetnarski and Ignaczak [8] provide a comprehensive review of these generalizations, highlighting the most notable advancements. Lord and Shulman [9] pioneered the theory of generalized thermoelasticity with one relaxation time (L-S theory), which represents the first significant generalization of the coupled theory. This theory addresses the limitation of infinite propagation speeds by replacing Fourier's law with the Maxwell-Cattaneo law. This modification introduces a relaxation time parameter, which accounts for the finite time required for thermal disturbances to propagate. The second significant generalization, often referred to as the G-L theory, involves the introduction of two relaxation times. This theory originated from Muller's [10] proposal of an entropy production inequality, which restricts the form of constitutive equations. Green and Laws [11] further refined this inequality, leading to explicit constitutive equations independently derived by Green and Lindsay [12] and Suhubi [13] Ignaczak's [14] comprehensive review offers a detailed analysis of both the L-S and G-L theories, along with key findings in the field.

Additionally, the concept of low-temperature thermoelasticity was introduced by Hetnarski and Ignaczak [15] (H-I theory). This model is distinguished by its nonlinear system of field equations, which sets it apart from the linear models discussed earlier. Due to its inherent complexity, the H-I theory will not be further explored in this work. Another generalization, proposed by Green and Naghdi [16] (G-N theory of type II), introduces a novel approach to thermoelasticity in the absence of energy dissipation. In contrast to the classical theory, this model replaces Fourier's law with a heat flux rate-temperature gradient relation. A distinctive feature of this theory is the absence of temperature rate terms in the heat equation, leading to undamped thermoelastic wave solutions. Additionally, Green and Naghdi [5], [16] further developed the G-N theory of type III, providing a more comprehensive framework for thermoelastic analysis. Finally, the dual-phase-lag thermoelasticity, was introduced by Tzou [17] (C-T theory). This theory departs from the traditional Fourier law by incorporating a modified heat conduction law with two distinct time delays, one for the heat flux and the other for the temperature gradient. This modification allows for a more accurate representation of heat conduction phenomena, particularly in cases involving rapid thermal transients and non-Fourier heat transport effects [18].

One of the most important uses of fractional derivatives in engineering, physics, and biology problems is the simulation of numerous natural phenomena, which is made possible by the fractional kinetic equations' capacity to capture the exponent of the diffusive substance's mean-squared displacement (see illustrative examples in [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Based on the fractional diffusion-wave equation, fractional thermoelasticity was taken into consideration in [30]. A quasi-static uncoupled model of thermoelasticity was examined in the initial Povstenko work, which only looked at thermally generated deformation and ignored mechanically induced thermal energy. During 2010–2011, several iterations of the fractional theory of thermoelasticity based on variations of the time-fractional Cattaneo equation were examined [31, 32, 33, 34]. A time-fractional version of the dual-phase-lag equation, which is used in the fractional theory of thermoelasticity, was examined in [35]. Additionally, a fractional version of the quasi-static theory of linked thermoelasticity was developed, which is based on a space-time fractional heat conduction law [36]. Subsequently, a mathematical model has been derived to describe the interaction of mechanical and thermal effects on a micropolar thermoelastic medium, utilizing the methodology of fractional calculus. The primary advantage of the fractional calculus model over the generalized model lies in its ability to predict a retarded response to these effects, which aligns more closely with observed natural phenomena, unlike the instantaneous response inherent to the generalized theory [37]. Furthermore, the occurrence condition for thermal resonance phenomenon during the electron–phonon interaction process in metals was investigated in [38], based on the hyperbolic two-temperature model.

This paper focuses on exploring the governing equations within the framework of quasi-static fractional thermoelasticity. We employed the space-time fractional Fourier law with anomalous thermal conductivity to model heat conduction processes. The initial value problem is formulated for an unbounded elastic domain, and the Laplace-Fourier transform technique is applied to simplify the governing field equations. Subsequently, we derive closed-form solutions for temperature and displacement in terms of the Fox H-function. Finally, we delve into numerical expansions for both temperature and displacement to gain further insights into the behaviour of the system. In section 2, the initial value problem on the unbounded elastic domain is formulated, the governing equations of the quasi-static theory of fractional thermoelasticity that employs the space-time fractional Fourier law with anomalous thermal conductivity are presented, and the field equations are simplified using the Laplace-Fourier transform technique. In section 3, we get closed-form solutions for displacement and temperature in terms of the Fox H-function that are valid for both the short-time and long-time domains. In section 4, we address the numerical expansions for displacement and temperature. Finally, in section 5, we provide a summary of the paper's key findings.

1. Mathematical problem

2.1. Governing equations

Within this sub-section, we are going to study the small changes in the shape and position of a thermoelastic solid caused by initial conditions related to both temperature and mechanical forces. These changes are known as affine infinitesimal deformations. We want to understand the equations that describe how the solid moves, considering only the forces caused by the material itself and ignoring any external forces like gravity or air resistance. These equations are based on the principle of conservation of momentum [39], [40]

Now by introducing the stress-strain constitutive relation as the following:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e \delta_{ij} - \chi_0 \theta \delta_{ij} \,, \tag{2.1}$$

where λ and μ are correspond to the standard Lamé constants, $\theta = T - T_0$ specifically θ is the temperature of the medium, T is the absolute temperature and T_0 is the temperature of the room, $\chi_0 = (3\lambda + 2\mu)\alpha_T$, clarifying that α_T is the parameter that quantifies the linear dimensional change of a material in response to temperature changing or known as the coefficient of linear thermal expansion, σ_{ij} are representing the components of Cauchy stress tensor, δ_{ij} is the Kronecker delta function. Beside the previous, u_i is the i-th component of displacement vector $\underline{u}$, $e = e_{ii} = u_{i,i} = e_{11} + e_{22} + e_{33}$ is known as the cubical dilation and e_{ij} is representing the strain tensor for linear elasticity defined that is define as:[41]

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2.2}$$

where the designated name for equation (2.2) is strain-displacement relation.

Now, we are going introduce also the equation of motion after establishing a zero-body force condition as $F_i = 0$ to get the following:

$$\sigma_{ij,j} = \rho \ddot{u}_i \,. \tag{2.3}$$

Consequently, equation (2.1) can be reformulated by using equation (2.2) as follows:

$$\sigma_{ij} = \mu\left(u_{i,j} + u_{j,i}\right) + \lambda u_{l,l}\delta_{ij} - \chi_0\theta\delta_{ij}\ , \tag{2.4}$$

and equation (2.4) can be rewritten by using equation (2.3) as follows:

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ij} - \chi_0 T_{,i} = \rho\ddot{u}_i\ . \tag{2.5}$$

Since the quasistatic assumption is allowed only when the rate of change of temperature and stress is sufficiently slow then; forces and wave propagation can be neglected. In cases of rapid change, the dynamic effects become so pronounced that the quasistatic assumption is rendered invalid. Now, we are going to adapt the assumption of quasi-static equilibrium, then equation (2.5) will take the form of:

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - \chi_0 T_{,i} = 0. \tag{2.6}$$

Now we introduce the thermal energy balance equation is given by the following:

$$-q_{i.i} + Q = \rho C_E\dot{T} + \chi_0 T_0\ \dot{e}\ , \tag{2.7}$$

in the absence of a heat source, we set $Q = 0$ and equation (2.7) can be rewritten as:

$$-q_{i.i} = \rho C_E\dot{T} + \chi_0 T_0\ \dot{e}\ , \tag{2.8}$$

And in the vector representation as follows:

$$-\boldsymbol{\nabla}\cdot\mathbf{q} = \rho C_E\dot{T} + \chi_0 T_0\ \dot{e}\ . \tag{2.9}$$

The classical Fourier law establishes a direct proportionality between the heat flux and the temperature gradient and denoted mathematically as:

$$q_i = -\kappa T_{,i}\ , \tag{2.10}$$

where κ is the thermal conductivity of the material, but we are going to replace the classical Fourier law (2.10) by the following rule: See [42], [43], [23]

$$q_i = -\kappa_{\alpha,\gamma}\ {}_{0}^{RL}\mathcal{D}_t^{1-\alpha}\ \frac{\partial^{\gamma-1}T}{\partial|x_i|^{\gamma-1}}. \tag{2.11}$$

where ${}_{a^+}^{RL}\mathcal{D}_t^{\alpha}$ is the left-sided Riemann-Liouville fractional derivative defined for a generic function $f(t),\ t\geq 0$ see [44], [45] and $\partial^{\alpha}g(x_i)/\partial|x_i|^{\alpha}$ is the Riesz fractional derivative defined for a generic function $g(x_i), x_i\in\mathbb{R}$. See [46]

In line with the suggestion proposed by Compte-Metzler for the anomalous diffusion coefficient we can assume the following functional forms for $\kappa_{\alpha,\gamma}$

represented by: [47]

$$\kappa_{\alpha,\gamma} = \kappa\tau^{1-\alpha}\ell_0^{\gamma-2}, \tag{2.12}$$

where κ is the classical thermal conductivity, τ is a characteristic time constant and ℓ_0 is also a characteristic length constant. τ and ℓ_0 will be specified subsequently. Equation (2.11) can be rewritten as the following:

$$q_i = -\kappa\ell_0^{\gamma-2}\tau^{1-\alpha}\,{}_0^{RL}\mathcal{D}_t^{1-\alpha}\frac{\partial^{\gamma-1}T}{\partial|x_i|^{\gamma-1}}. \tag{2.13}$$

Now the governing equation for energy balance in one dimension representation is expressed as the following:

$$\rho C_E\dot{T} + \chi_0 T_0\,\dot{e} = -\frac{\partial q}{\partial x}, \tag{2.14}$$

where $\dot{e} = \frac{\partial^2 u}{\partial x\partial t}$. We can proceed by eliminating the heat flux term that appears in the energy balance equation (2.14) and equation (2.13) by employing a suitable technique as follow:

$$\rho C_E\dot{T} + \chi_0 T_0\,\dot{e} = \kappa\ell_0^{\gamma-2}\tau^{1-\alpha}\,{}_0^{RL}\mathcal{D}_t^{1-\alpha}\frac{\partial^{\gamma}T}{\partial|x_i|^{\gamma}}, \tag{2.15}$$

where

$$\frac{\partial}{\partial x}\left[\frac{\partial^{\gamma-1}}{\partial(x)^{\gamma-1}}T(x,t)\right] = \frac{\partial^{\gamma}T}{\partial|x_i|^{\gamma}},$$

2.2. Problem setting up.

Within this section, we are going to establish the mathematical framework governing the system that is previously mentioned and described by equations (2.6) and (2.15), defined across an unbounded spatial domain as $-\infty < x < \infty$. As a preliminary step we are presenting the initial conditions that govern the problem as follows:

$$\begin{gathered} T(x,0) = T_0 + \vartheta_0\delta(x), \\ \sigma_{xx}(x,0) = 0. \end{gathered} \tag{2.16}$$

While the stress initial condition specified in equation (2.16) is conventionally applied to the displacement itself, more examination reveals that equation (2.15) necessitates the imposition of this condition on the displacement gradient instead, we insert a comprehensive physical constraint that specifies the initially stress-free equilibrium state of the medium. Such an initial condition is considered appropriate when the deformation process occurs at a sufficiently slow pace, allowing the system to be considered in a quasi-static equilibrium state at each instant. Certainly, the principle of momentum conservation dictates that $\sigma_{ji,j} = 0$, this expression can be reduced to a one-dimensional form as follows

$$\begin{gathered}\frac{\partial \sigma_{xx}(x,t)}{\partial x} = 0 \\ \sigma_{xx}(x,t) = \sigma_0(t).\end{gathered} \tag{2.17}$$

Within the infinite spatial regions, the stress tensor exhibits symmetric properties. Moreover, it is convenient that the stress diminishes to negligible levels as the spatial coordinates tend towards infinity. Consequently, the function representing the stress component $\sigma_0(t)$, can be reasonably approximated to zero in these limiting conditions.

Consequently, we can adopt the initial condition where the displacement vector is identically zero as $\sigma_{xx}(x,0) = 0$. Moreover, the component of the normal stress tensor to any surface within the domain is found to be null at all points as the following:

$$\sigma_{xx}(x,t) = 0. \tag{2.18}$$

Recalling the specific initial conditions outlined in equation (2.16), the complexity of the problem can be reduced to a one-dimensional framework. This simplification arises from the nature of the initial state, which allows a focused analysis along a single spatial dimension. Meaning that all physical quantities involved in the problem can be expressed as functions of the spatial variable x (one dimension setting) and time t. Consequently, the governing equations represented by equations (2.6) and (2.15) can be simplified to involve only derivatives with respect to these two independent variables as the following:

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} = \chi_0 \frac{\partial T}{\partial x}, \tag{2.19}$$

$$\begin{gathered}\rho C_E \frac{\partial T}{\partial t} + \chi_0 T_0 \, \frac{\partial^2 u}{\partial x \partial t} = \\ \kappa \ell_0^{\gamma-2} \tau^{1-\alpha} \, {}_0^{RL}\mathcal{D}_t^{1-\alpha} \frac{\partial^\gamma T}{\partial |x|^\gamma}.\end{gathered} \tag{2.20}$$

Considering the constitutive relation provided in equation (2.1) and the previously established one-dimensional framework, the individual components of the Cauchy stress tensor can be explicitly determined by the following:

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \chi_0(T - T_0), \tag{2.21}$$

$$\sigma_{yy} = \sigma_{zz} = \lambda\frac{\partial u}{\partial x} - \chi_0(T - T_0), \tag{2.22}$$

The hydrostatic stress σ_H which known as a fundamental quantity in continuum mechanics, that is defined as the arithmetic mean of the normal components of the stress tensor. Mathematically, as the following:

$$\sigma_H = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{2}{3}\left[\lambda\frac{\partial u}{\partial x} - \chi_0(T - T_0)\right]. \tag{2.23}$$

By incorporating the previously established relationship in equation (2.18), the initial conditions early presented in equation (2.16) can be reformulated. This reformulation is gained by substituting the expressions from equation (2.18) and the subsequent definition in equations (2.21) and (2.22) into the original initial condition equations as the following:

$$\begin{aligned} &T(x,0) = T_0 + \vartheta_0\delta(x), \\ &\frac{\partial u(x,0)}{\partial x} = \frac{\chi_0\vartheta_0}{\lambda + 2\mu}\delta(x), \end{aligned} \tag{2.24}$$

by performing integration on the second initial condition specified in equation (2.24) across the designated spatial interval $(-\infty, 0]$. Assuming a prescribed initial displacement state is given as $u(-\infty, 0) = u_{-\infty}$, we get the following:

$$\begin{aligned} &T(x,0) = T_0 + \vartheta_0\delta(x), \\ &u(x,0) = u_{-\infty} + \frac{\chi_0\vartheta_0}{\lambda + 2\mu}\begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0,. \\ 0, & x < 0. \end{cases} \end{aligned} \tag{2.25}$$

Now, we are going to introduce new dimensionless variables, we can reduce the number of independent parameters, making the problem easier to analyse and understand. This process considered valuable when we deal with equations that involve multiple units or when we want to compare results across different systems or scales.

The following transformations involve combining existing variables with suitable constants or physical quantities to create new dimensionless quantities as the following:

$$x=\frac{x}{c_1\eta},\ \ u=\frac{u}{c_1\eta},\ \ u_{-\infty}=\frac{U_{-\infty}}{c_1\eta}\ \ \ell_0=\frac{\ell_0}{c_1\eta},\ \ t=\frac{t}{c_1^2\eta},$$

$$\tau=\frac{\tau}{c_1^2\eta},\ \ (\sigma_{ij},\sigma_H)=(\lambda+2\mu)(\sigma_{ij},\sigma_H),\eta=\frac{\rho C_E}{\kappa},\qquad(2.26)$$
$$T=\frac{\lambda+2\mu}{\chi_0}\theta+T_0,\ \ c_1^2=\frac{\lambda+2\mu}{\rho},$$

therefore, the governing equations (2.19), (2.20) can be rewritten as the following:

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial\theta}{\partial x}\ ,\qquad(2.27)$$

and

$$\frac{\partial\theta}{\partial t}+\varepsilon\frac{\partial^2 u}{\partial x\partial t}=\ell_0^{\gamma-2}\tau^{1-\alpha}\,{}_0^{RL}\mathcal{D}_t^{1-\alpha}\frac{\partial^\gamma\theta}{\partial|x|^\gamma}\ ,\qquad(2.28)$$

where

$$\varepsilon=\frac{{\chi_0}^2T_0}{\rho C_E(\lambda+2\mu)}\ .$$

Additionally, the constitutive relations (2.18), (2.21), (2.22) and (2.23) can be also rewritten as the following:

$$\sigma_{xx}=\frac{\partial u}{\partial x}-\theta,$$
$$\sigma_{yy}=\sigma_{zz}=\frac{\lambda}{\lambda+2\mu}\frac{\partial u}{\partial x}-\theta.\qquad(2.29)$$

Additionally, we can get the following:

$$\sigma_H=\frac{2}{3}(\delta_0-1)\theta,\qquad(2.30)$$

where

$$\delta_0=\frac{\lambda}{\lambda+2\mu}.$$

and the initial condition (2.25) can be rewritten as the following:

$$\theta(x,0)=\Theta_0\delta(x),\ \ \frac{\partial u(x,0)}{\partial x}=\Theta_0\delta(x),$$
$$u(x,0)=U_{-\infty}+\Theta_0\begin{cases}1, & x>0,\\ \frac{1}{2}, & x=0,\\ 0, & x<0.\end{cases} \tag{2.31}$$

where

$$\Theta_0=\frac{\chi_0\vartheta_0 c_1\eta}{\lambda+2\mu}.$$

Furthermore, we have incorporated the relationship derived from equations (2.18), (2.21), (2.22) and (2.26) $\partial u/\partial x=\theta$ into our analysis. This relationship has proven instrumental in the derivation of the hydrostatic stress expression presented in equation (2.30).

2.3. Solution in the transformed domain.

In this sub-section, we will derive an analytical solution for the temperature field within the Laplace-Fourier domain. Subsequently, we will investigate the crucial question of whether the non-negative nature of the temperature distribution is maintained even when thermomechanical coupling effects are taken into consideration. Now we are going to apply Laplace transform on equations (2.27) and (2.28) as the following:

$$\frac{\partial^2\tilde{u}}{\partial x^2}=\frac{\partial\tilde{\theta}}{\partial x}\,,$$
$$s\tilde{\theta}-\Theta_0\delta(x)+\varepsilon s\frac{\partial\tilde{u}}{\partial x}-\varepsilon\Theta_0\delta(x) \tag{2.32}$$
$$=\ell_0^{\gamma-2}\tau^{1-\alpha}s^{1-\alpha}\frac{\partial^\gamma\tilde{\theta}}{\partial|x|^\gamma}.$$

Consequently, we apply Fourier transform on the previous set of equations (2.32) as the following:

$$-q^2\hat{\tilde{u}}=iq\,\hat{\tilde{\theta}}\,,$$
$$s\,\hat{\tilde{\theta}}(q,s)-\Theta_0+iq\varepsilon s\hat{\tilde{u}}(q,s)-\varepsilon\Theta_0 \tag{2.33}$$
$$=-\ell_0^{\gamma-2}\tau^{1-\alpha}s^{1-\alpha}|q|^\gamma\hat{\tilde{\theta}}(q,s)\,,$$

by employing the first part into the second part of (2.33), as $\hat{\tilde{\theta}} = iq\hat{\tilde{u}}$ we get the following:

$$\hat{\tilde{\theta}}(q,s) = \frac{(1+\varepsilon)\Theta_0}{(1+\varepsilon)s + \ell_0^{\gamma-2}\tau^{1-\alpha}s^{1-\alpha}|q|^{\gamma}}, \tag{2.34}$$

As a reminder, the tilde symbol ($\tilde{}$) represents the Laplace transform, which is a mathematical operation that transforms a function of time into a function of a complex variable denoted by s. The hat symbol ($\hat{}$) represents the Fourier transform, which is another mathematical operation that transforms a function of time into a function of a frequency parameter denoted by q.

One of the important questions about the temperature resulting from the thermomechanical coupling (2.34) concerns its nonnegativity and the conservation of thermal energy. We consider the special case $\gamma = 2$, and invert the Fourier transform in (2.34), we obtain the following:

$$\hat{\tilde{\theta}}(q,s) = \frac{\frac{(1+\varepsilon)\Theta_0}{\tau^{1-\alpha}s^{1-\alpha}}}{\frac{(1+\varepsilon)s}{\tau^{1-\alpha}s^{1-\alpha}} + |q|^2},$$

$$\tilde{\theta}(x,s) = \mathcal{F}^{-1}\left\{\frac{\frac{(1+\varepsilon)\Theta_0}{\tau^{1-\alpha}s^{1-\alpha}}}{\frac{(1+\varepsilon)s}{\tau^{1-\alpha}s^{1-\alpha}} + |q|^2}\right\} = \frac{\Theta_0\sqrt{1+\varepsilon}}{2\sqrt{s\psi_0(s)}}\exp\left(-\sqrt{1+\varepsilon}\sqrt{\frac{s}{\psi_0(s)}}|x|\right) \tag{2.35}$$

where

$$\psi_0(s) = \tau^{1-\alpha}s^{1-\alpha}.$$

Considering equation (2.35), we observe that the solution obtained in the Laplace domain exhibits a similar form to that encountered in the context of anomalous thermal diffusion whenever we set $\varepsilon = 0$. Additionally, the inclusion of the factor $\sqrt{1+\varepsilon}$ both within and outside the exponential function does not alter the fundamental characteristic of temperature as a completely monotone function being a product of two completely monotone functions as the following:

$$\tilde{\theta}(x,s) = \frac{1}{2}g_1(\lambda)g_2(\lambda),$$

where

$$g_1(\lambda) = \frac{1}{\sqrt{\lambda\psi_0(\lambda)}}\,, g_2(\lambda) = \exp\left(-\sqrt{1+\varepsilon}\sqrt{\frac{\lambda}{\psi_0(\lambda)}}\right),$$

clearly,

$$g_1(\lambda) = \frac{1}{\sqrt{\lambda\psi_0(\lambda)}} \in SF \subseteq CMF,$$

and since $e^{|x|\lambda} \in CMF$ then; $g_2(\lambda) = e^{-\sqrt{1+\varepsilon}\sqrt{\frac{\lambda}{\psi_0(\lambda)}}} \in CMF$. Therefore, $g_1(\lambda)g_2(\lambda) \in CMF$ which leads to $\tilde{\theta}(x,s) \in CMF$ when the Laplace parameter is defined on the positive real axis.

Therefore, the property of temperature $\theta(x,t)$ being always positive (non-negative) for $\mathbb{R} \times \mathbb{R}_+ \cup \{0\}$ is maintained even when considering quasi-static thermoelasticity that employs the modified bi-fractional Fourier constitutive law that defined by equation (2.13).

It is important to note that when analysing the dynamic theory, which takes into consideration the force caused by inertia $\rho\ddot{u}$, the non-negativity of temperature must be carefully considered. To determine the temperature distribution, we can integrate both sides of equation (2.35) over the appropriate range and then apply the inverse Laplace transform as the following:

$$\int_{-\infty}^{\infty} \tilde{\theta}(x,s)dx = \frac{\Theta_0\sqrt{1+\varepsilon}}{2\sqrt{s\psi_0(s)}}\int_{-\infty}^{\infty} e^{-\sqrt{1+\varepsilon}\sqrt{\frac{s}{\psi_0(s)}}|x|}\,dx = \frac{\Theta_0}{s},$$

finally, we get

$$\int_{-\infty}^{\infty} \theta(x,t)\,\mathrm{d}x = \Theta_0\mathcal{H}(t), \tag{2.36}$$

where the Heaviside unit step function denoted by $\mathcal{H}(t)$, is a mathematical function that abruptly changes from 0 to 1 at a specific time, typically $t = 0$. It is widely used to model sudden changes in quantities, like the initial temperature distribution in a system. [48]

The initial temperature condition in thermoelasticity can be represented by the Heaviside function. The Heaviside function can be used to express this discontinuity if the temperature is originally zero everywhere except at the interfacial boundary, where it equals Θ_0.

By calculating the total thermal energy using equation (2.36) and comparing it to this initial temperature distribution, we can assess whether thermal energy is conserved within the system. If the total thermal energy remains constant over time, it indicates that thermal energy is indeed conserved.

A simplified model known as the quasi-static theory of thermoelasticity assumes that the body's deformation is sufficiently gradual to be regarded as equilibrium at every instant in time. This simplification allows for easier analysis of the system's behaviour. The modified bi-fractional Fourier constitutive law extends the classical Fourier law of heat conduction by incorporating non-integer order derivatives in the heat flux equation. This generalization enables modelling materials with anomalous diffusion properties. A more realistic depiction of the material's thermal behaviour can be achieved by integrating this modified law into the quasi-static theory of thermoelasticity. In summary, by comparing the calculated total thermal energy to the initial temperature distribution, we can determine that thermal energy is conserved in the system when the quasi-static theory of thermoelasticity is employed in conjunction with the modified bi-fractional Fourier constitutive law (2.13). This approach provides a more comprehensive and accurate understanding of the material's thermal properties.

2. Closed-form solutions

within this section of our discussion, we are going to develop specific mathematical equations to calculate the temperature, hydrostatic stress, and displacement within the material. These equations will be based on a particular model of heat diffusion, known as the bi-fractional diffusion equation. This model is simplified by neglecting the effects related to the space fractality of the heat namely, $\gamma = 2$. To ensure a more comprehensive solution, we are going to introduce an equation that allows us to calculate the temperature. To begin this process, we are going to take equation (2.34) and rewrite it in a different but equivalent format.

$$\hat{\bar{\theta}}(q,s) = \frac{\Theta_0 s^{\alpha-1}}{s^{\alpha} + \dfrac{\ell_0^{\gamma-2}\tau^{1-\alpha}|q|^{\gamma}}{(1+\varepsilon)}}\ , \tag{3.1}$$

to convert the function from the (s-domain) Laplace domain, back to the time domain (t-domain) we use the inverse Laplace transform as the following: [49]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+C}\right\} = E_{\alpha}(-ct^{\alpha}),\ \ C > 0. \tag{3.2}$$

Therefore,

$$\tilde{\theta}(q,t) = \mathcal{L}^{-1}\left\{\frac{\Theta_0 s^{\alpha-1}}{s^{\alpha}+\dfrac{\ell_0^{\gamma-2}\tau^{1-\alpha}|q|^{\gamma}}{(1+\varepsilon)}}\right\} = \Theta_0 E_{\alpha}\left(-\frac{\ell_0^{\gamma-2}\tau^{1-\alpha}|\omega|^{\gamma}}{(1+\varepsilon)}t^{\alpha}\right). \tag{3.3}$$

where the symbol $E_{\alpha}(\cdot)$ denotes the Mittag-Leffler function with one parameter α, as defined in reference [50] . By applying the inverse Fourier transform to equation (3.3), referencing the relationships (A.10) and (A.20) in [19], we can derive a closed-form expression for temperature, as follows:

$$\tilde{\theta}(q,t) = \Theta_0 H_{1,2}^{1,1}\left[\frac{\ell_0^{\gamma-2}\tau^{1-\alpha}|\omega|^{\gamma}}{(1+\varepsilon)}t^{\alpha}\middle|\begin{matrix}(0,1)\\(0,1),(0,\alpha)\end{matrix}\right]. \tag{3.4}$$

Upon applying the inverse Fourier transformation to equation (3.4), we arrive at the following outcome:

$$\theta(x,t) = \frac{\Theta_0}{\sqrt{4\pi}}\left(\frac{1+\varepsilon}{\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{1}{\gamma}} \times \\ \times H_{2,3}^{2,1}\left[\frac{(1+\varepsilon)|x|^{\gamma}}{2^{\gamma}\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\middle|\begin{matrix}\left(1-\frac{1}{\gamma},1\right);\left(1-\frac{\alpha}{\gamma},\alpha\right)\\\left(0,\frac{\gamma}{2}\right),\left(1-\frac{1}{\gamma},1\right);\left(\frac{1}{2},\frac{\gamma}{2}\right)\end{matrix}\right]. \tag{3.5}$$

The symbol $H_{p,q}^{m,n}[\cdot]$ represents the Fox H-function, a mathematical function defined in terms of the Mellin-Barnes integral, as outlined in reference [51].

When examining the solution represented by equation (3.5), we notice that the closed-form expression of this solution can be modified or transformed. This implies that the mathematical formula describing this solution can be altered through various mathematical operations or techniques. A notable feature of the closed-form solution presented in equation (3.5) is the inclusion of the thermomechanical coupling parameter $\varepsilon \neq 0$, this parameter takes into account the impact of mechanical effects on the development of temperature. In the situation where these mechanical effects are minimal, meaning that ε is negligible, equation (3.5) can be simplified to the following form:

$$\theta(x,t) = \frac{\Theta_0}{\sqrt{4\pi}}\left(\frac{1}{\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{1}{\gamma}} \times \\ \times H_{2,3}^{2,1}\left[\frac{|x|^{\gamma}}{2^{\gamma}\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\middle|\begin{matrix}\left(1-\frac{1}{\gamma},1\right);\left(1-\frac{\alpha}{\gamma},\alpha\right)\\ \left(0,\frac{\gamma}{2}\right),\left(1-\frac{1}{\gamma},1\right);\left(\frac{1}{2},\frac{\gamma}{2}\right)\end{matrix}\right]. \tag{3.6}$$

On the other hand, it is important to note that the hydrostatic stress denoted by σ_H, is directly proportional to the temperature solution. The displacement represented by the symbol $u(x,t)$, can be calculated from the normal stress component $\sigma_{xx}(x,t)$. By utilizing the dimensionless transformations defined in equation (2.26) and applying them to equations (2.18), (2.21), and (2.22), we can derive a relationship between the gradient of the displacement $\frac{\partial u}{\partial x}$, and the temperature θ. This relationship is given by the following:

$$\frac{\partial u(x,t)}{\partial x} = \theta(x,t). \tag{3.7}$$

By integrating both sides of (3.7) on the interval $(-\infty, x)$, so we get the following:

$$u(x,t) = u(-\infty,t) + \int_{-\infty}^{x}\theta(\xi,t)\,\mathrm{d}\xi. \tag{3.8}$$

Before evaluating the integral shown in equation (3.8), it is crucial to establish a specific boundary condition for the displacement at $x \to -\infty$. Considering the symmetrical nature of the temperature distribution (an even function), we can deduce that:

$$\int_{-\infty}^{0}\theta(\xi,t)\,\mathrm{d}\xi = \int_{0}^{\infty}\theta(\xi,t)\,\mathrm{d}\xi = \mathcal{M}\{\theta(x,t),x\}(1,t), \tag{3.9}$$

The relation $\mathcal{M}\{f(t)\}(z) = \int_0^{\infty} t^{z-1}f(t)\,\mathrm{d}t$ represents the Mellin transform of the function $f(t)$, where $t > 0$ and the Mellin transform of the Fox H-function is given as the following:

$$\int_0^{\infty} x^{z-1}H_{p,q}^{m,n}\left[ax\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]dx \\ = a^{-z}\frac{\left[\Pi_{j=1}^{m}\Gamma\left(b_j+B_jz\right)\right]\left[\Pi_{j=1}^{n}\Gamma\left(1-a_j+A_jz\right)\right]}{\left[\Pi_{j=m+1}^{q}\Gamma\left(1-b_j-B_jz\right)\right]\left[\Pi_{j=n+1}^{p}\Gamma\left(a_j-A_jz\right)\right]}, \tag{3.10}$$

therefore, after using the relation mentioned in p.12 in [51] and noticing that $z = 1$ in (3.10) the integral expression presented in equation (3.9) can be transformed or simplified into the following form:

$$\int_{-\infty}^{0} \theta(\xi, t)\, \mathrm{d}\xi = \int_{0}^{\infty} \theta(\xi, t)\, \mathrm{d}\xi = \frac{\Theta_0}{\gamma} \mathcal{H}(t). \tag{3.11}$$

When calculating the integral represented by equation (3.11), it is crucial to keep in mind that this integral appears to be inconsistent with the principle of conservation of thermal energy, particularly when the thermomechanical coupling parameter $\gamma = 2$ that is not equal to zero. This inconsistency becomes evident when comparing equation (3.11) with equation (3.9), which expresses the conservation of thermal energy. To address this issue, equation (2.37) can be extended to encompass a broader range of scenarios as follows:

$$\int_{-\infty}^{\infty} \theta(\xi, t)\, \mathrm{d}\xi = \frac{2\Theta_0}{\gamma} \mathcal{H}(t). \tag{3.12}$$

By revisiting equation (3.8), we can express the given information in the following system of equations:

$$\begin{aligned} u(\infty, t) &= u(-\infty, t) + \frac{2\Theta_0}{\gamma} \mathcal{H}(t), \\ u(0, t) &= u(-\infty, t) + \frac{\Theta_0}{\gamma} \mathcal{H}(t), \end{aligned} \tag{3.13}$$

The equations presented in the preceding statement demonstrate that the boundary condition for displacement, denoted by $u(\infty, t)$ and the interfacial boundary condition $u(0, t)$, which characterizes the interaction between the material and its surroundings, are dependent on the specific selection of the displacement field $x \to -\infty$. Moreover, by analysing the derived initial conditions $u(x, 0)$, as outlined in equation (2.31), we conclude that there exists an infinite number of feasible choices for the boundary condition $u(-\infty, t)$. This plethora of possibilities compels us to select one particular boundary condition as:

$$u(-\infty, t) = U_{-\infty} \mathcal{H}(t), \tag{3.14}$$

which agrees with the derived initial condition in (2.31), $u(-\infty, 0) = U_{-\infty}$. Consequently, we have from (3.13) and (3.14) that

$$\begin{aligned} u(\infty, t) &= \left(U_{-\infty} + \frac{2\Theta_0}{\gamma}\right) \mathcal{H}(t), \\ u(0, t) &= \left(U_{-\infty} + \frac{\Theta_0}{\gamma}\right) \mathcal{H}(t). \end{aligned} \tag{3.15}$$

As we encountered a similar issue of thermal energy conservation in the context of the space fractality equation (3.12), we observe an inconsistency when comparing the derived initial conditions $u(x,0)$, obtained using the specified initial condition on temperature $\theta(x,0)=\Theta_0\mathcal{H}(t)$, with the derived boundary conditions on $u(\infty,t)$ and $u(0,t)$, obtained using the temperature solution (3.5). This discrepancy stems from the differing calculation methods employed. However, it is important to note that this inconsistency disappears as the space fractality parameter γ approaches 2, indicating the absence of space fractality.

By incorporating the imposed boundary condition from equation (3.14) and the integral expression from equation (3.11), we can recast the displacement equation (3.8) into the following equivalent form:

$$u(x,t)=\left(U_{-\infty}+\frac{\Theta_0}{\gamma}\right)\mathcal{H}(t)+\int_0^x\theta(\xi,t)\,\mathrm{d}\xi,\quad x\in\mathbb{R}. \tag{3.16}$$

Now by using the Euler transform of the Fox H-function see p.12 and p. 59 in [51].

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}H_{p,q}^{m,n}\left[bx^k\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]\mathrm{d}x$$
$$=t^{\rho+\sigma-1}\Gamma(\sigma)H_{p+1,q+1}^{m,n+1}\left[bt^k\left|\begin{matrix}(1-\rho,k),(a_p,A_p)\\(b_q,B_q),(1-\rho-\sigma,k)\end{matrix}\right.\right],$$

in our situation we are going to use $\rho=1,\sigma=1$ and $k=1$ into the previous equation of Euler transform, then the integral in equation (3.16) can be calculated on the following closed-form:

$$u(x,t)=\left(U_{-\infty}+\frac{\Theta_0}{\gamma}\right)\mathcal{H}(t)+\frac{\Theta_0}{\gamma\sqrt{4\pi}}\operatorname{sign}(x)|x|\left(\frac{1+\varepsilon}{\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{1}{\gamma}}$$
$$\times H_{3,4}^{2,2}\left[\left(\frac{1+\varepsilon}{2^{\gamma}\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{1}{\gamma}}|x|\left|\begin{matrix}(0,1),\left(1-\frac{1}{\gamma},1\right);\left(1-\frac{\alpha}{\gamma},\alpha\right)\\\left(0,\frac{\gamma}{2}\right),\left(1-\frac{1}{\gamma},1\right);\left(\frac{1}{2},\frac{\gamma}{2}\right),(-1,1)\end{matrix}\right.\right], \tag{3.17}$$

3. Discussion

In this section, we will examine the closed-form expression that we derived in the previous section. These expressions represent the temperature, stresses and displacement within the material. To better understand these expression, we will utilize the series expansion of the Fox H-function, which can be found on page 218 of the reference [51]. By applying the series expansion of the Fox H-function to the temperature solution represented by equation (3.5), we obtain the following expression

$$\theta(x,t)=\frac{\Theta_0}{\sqrt{4\pi}}\left(\frac{1+\varepsilon}{\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{1}{\gamma}}\{\theta_1(x,t)+\theta_2(x,t)\}, \tag{4.1}$$

to obtain $\theta_1(x,t)$, we are going to use $b_h=b_1=0$, $B_h=B_1=\frac{\gamma}{2}$ and $\frac{b_1+\nu}{B_1}=\frac{2\nu}{\gamma}$ then we get the following:

$$\theta_1(x,t)=\frac{2}{\gamma}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\left(\frac{(1+\varepsilon)|x|^{\gamma}}{2^{\gamma}\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{2\nu}{\gamma}}\frac{\Gamma\left(1-\frac{1}{\gamma}-\frac{2\nu}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}+\frac{2\nu}{\gamma}\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(1-\frac{\alpha}{\gamma}-\frac{2\alpha\nu}{\gamma}\right)}, \tag{4.2}$$

Similarly, for $\theta_2(x,t)$ we use $b_h=b_2=1-\frac{1}{\gamma}$, $B_h=B_2=1$ and $\frac{b_2+\nu}{B_2}=1-\frac{1}{\gamma}+\nu$.

then we get the following:

$$\theta_2(x,t)=\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\left(\frac{(1+\varepsilon)|x|^{\gamma}}{2^{\gamma}\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{1-\frac{1}{\gamma}+\nu}\frac{\Gamma\left(\frac{1}{2}-\frac{\gamma}{2}(1+\nu)\right)\Gamma(1+\nu)}{\Gamma\left(\frac{\gamma}{2}(1+\nu)\right)\Gamma\big(1-\alpha(1+\nu)\big)}. \tag{4.3}$$

Finally, by once again applying the series expansion to the closed-form expression of displacement (3.17), we arrive at the following result

$$\begin{aligned}u(x,t)&=\left(U_{-\infty}+\frac{\Theta_0}{\gamma}\right)\mathcal{H}(t)\\&+\frac{\Theta_0}{\gamma\sqrt{4\pi}}\operatorname{sign}(x)|x|\left(\frac{1+\varepsilon}{\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{1}{\gamma}}\{u_1(x,t)+u_2(x,t)\},\end{aligned} \tag{4.4}$$

to get $u_1(x,t)$ we are going to use the substitution into the series expansion as $b_h=b_1=0$, $B_h=B_1=\frac{\gamma^2}{2}$ and $\frac{b_1+\nu}{B_1}=\frac{2\nu}{\gamma^2}$, then we get the following:

$$u_1(x,t)=2\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!\,(1+2\nu)}\left(\frac{(1+\varepsilon)|x|^{\gamma}}{2^{\gamma}\ell_0^{\gamma-2}\tau^{1-\alpha}t^{\alpha}}\right)^{\frac{2\nu}{\gamma}}\frac{\Gamma\left(1-\frac{1}{\gamma}-\frac{2\nu}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}+\frac{2\nu}{\gamma}\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(1-\frac{\alpha}{\gamma}-\frac{2\alpha\nu}{\gamma}\right)} \tag{4.5}$$

and similarly, to get $u_2(x,t)$ we are going to use into the series expansion as $b_h = b_2 = 1 - \frac{1}{\gamma}$, $B_h = B_2 = \gamma$ and $\frac{b_2+\nu}{B_2} = \frac{1-\frac{1}{\gamma}+\nu}{\gamma}$, then we get the following:

$$u_2(x,t) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(1+\nu)!} \left(\frac{(1+\varepsilon)|x|^\gamma}{2^\gamma \ell_0^{\gamma-2} \tau^{1-\alpha} t^\alpha} \right)^{1-\frac{1}{\gamma}+\nu} \frac{\Gamma\left(\frac{1}{2} - \frac{\gamma}{2}(1+\nu)\right)\Gamma(1+\nu)}{\Gamma\left(\frac{\gamma}{2}(1+\nu)\right)\Gamma\big(1-\alpha(1+\nu)\big)} \quad (4.6)$$

For our numerical calculations, we have selected copper as our material and then, we are going to use the specific values of its physical properties as measured at room temperature $300^0\mathrm{K}$. These values will be used in our mathematical models to simulate the behaviour of the copper under different conditions.

$$\begin{aligned} &\varepsilon = 0.0168, \ \ \delta_0 = 0.5013, \ \ c_1\eta = 3.695 \times 10^7 \mathrm{m}^{-1}, \\ &\quad c_1^2\eta = 1.536 \times 10^{11} \mathrm{sec}^{-1}. \end{aligned} \quad (4.7)$$

Since $\ell_0 = 1/c_1\eta$ and $\tau = 1/c_1^2\eta$, respectively, the dimensional characteristic constantsℓ_0 and τ are selected so that the dimensionless constants in all closed-form expressions and their series expansion equal unit. Furthermore, we assign the dimensionless constants $U_{-\infty}$ and Θ_0 to equal one, specifically:

$$\ell_0 = 1, \ \ \tau = 1, \ \ \Theta_0 = 1, \ \ U_{-\infty} = 1. \quad (4.8)$$

Unless we explicitly state otherwise, we will use the following specific values for the material properties in our calculations. These values represent the standard or default parameters that we will assume for the material unless we provide different data.

$$\alpha = 0.30001, \ \ \gamma = 1.799. \quad (4.9)$$

Equations (4.1)–(4.3) are the mathematical formulae that we directly applied in a numerical technique to determine the temperature, displacement, and hydrostatic stress. We approximated the infinite series involved in these formulas by adding up the seventy-first terms, meaning that the summations are carried out across the range of 0 to 70. Using its series representation (4.1)-(4.3), we graphically depict the temperature distribution in Fig. 1 or the solution provided by equations (3.5). Equation (2.30) in this case shows a linear relationship between the temperature and the hydrostatic stress. We examine how the space fractality γ and the thermoelastic coupling ε affect the hydrostatic stress in Fig. 2. We choose two different values for the thermoelastic coupling $\varepsilon = 0.0168$ and $\varepsilon = 0.5$

based on its definition $\varepsilon = \frac{\gamma^2 T_0}{\rho C_E(\lambda+2\mu)}$, where it depends on the room temperature T_0 besides the material parameters. Fig. 2, suggests that an increase in hydrostatic stress close to the interfacial boundary surface is effectively caused by an increase in the thermoelastic coupling. Furthermore, two distinct space fractional parameters $\gamma = 1.6999 \cong 1.7$ and $\gamma = 2.0001 \cong 2.$ are used to represent the hydrostatic stress. Avoiding gamma function singularities in the denominators of the expansions (4.1)-(4.3) and (4.4)-(4.6) is the primary reason for using such fractional parameter values (4.9). In view of Fig.2, it is evident that the use of two different space fractional parameters results in an apparent numerical effect on the hydrostatic stress curve. The use of two different space fractional values results in a discernible numerical artifact in the hydrostatic stress curve, as seen in Fig. 2(b). The differences shown in the regions beneath each curve show the main influence of the space fractional parameter on the temperature and hydrostatic stress profiles. In particular, the following is a mathematical expression for the area beneath the hydrostatic stress curve: [52]

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \sigma_H(\xi,t)\,\mathrm{d}\xi \right| &= \frac{2}{3}|\delta_0 - 1| \int_{-\infty}^{\infty} \theta(\xi,t)\,\mathrm{d}\xi \\ &= \frac{4\Theta_0|\delta_0 - 1|}{3\gamma}\mathcal{H}(t) = \frac{0.6649}{\gamma}, \\ & t > 0. \end{aligned} \tag{4.10}$$

The area beneath the hydrostatic stress curve is influenced by the selection of the space fractional parameter γ. Specifically, this area varies with the value of the space fractional parameter as $\left|\int_{-\infty}^{\infty} \sigma_H(\xi,t)\,\mathrm{d}\xi\right| = 0.39114$ where $\gamma = 1.6999$ and $\left|\int_{-\infty}^{\infty} \sigma_H(\xi,t)\,\mathrm{d}\xi\right| = 0.3325$ corresponds to the absence of space fractality. In Fig.3, we examine how the thermal conductivity's temporal crossover affects the material's thermally induced displacement. As it coincides well with the displacement caused by a fixed thermal conduction in the short-time domain and behaves similarly to the displacement caused by a different fixed thermal conduction in the long-time domain, it is evident that the displacement exhibits similar crossover behaviour. In the long-time domain, it seems as though the displacement reaches deeper regions within the material. Additionally, we examine the impact of a time-dependent crossover in thermal conductivity on the material's thermally induced displacement in Fig. 4. We also investigate how space fractality and thermoelastic coupling affect the displacement at a particular moment in time.

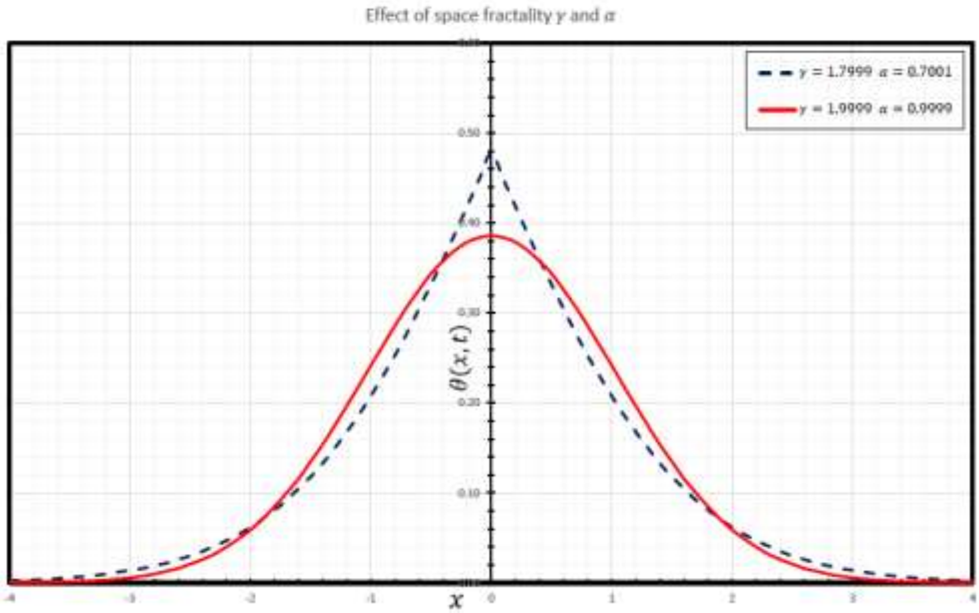

Figure 1: Analysis demonstrates the influence of spatial fractality γ and the parameter α on temperature distribution. The first series shows the temperature solution with $\alpha = 0.70001$ and $\gamma = 1.7999$ and the second one shows the temperature solution with $\alpha = 0.999999$ and $\gamma = 1.9999$ i.e. α approaches 1 and spatial fractality approaches 2, noticing that the temperature solution converges towards a normal distribution.

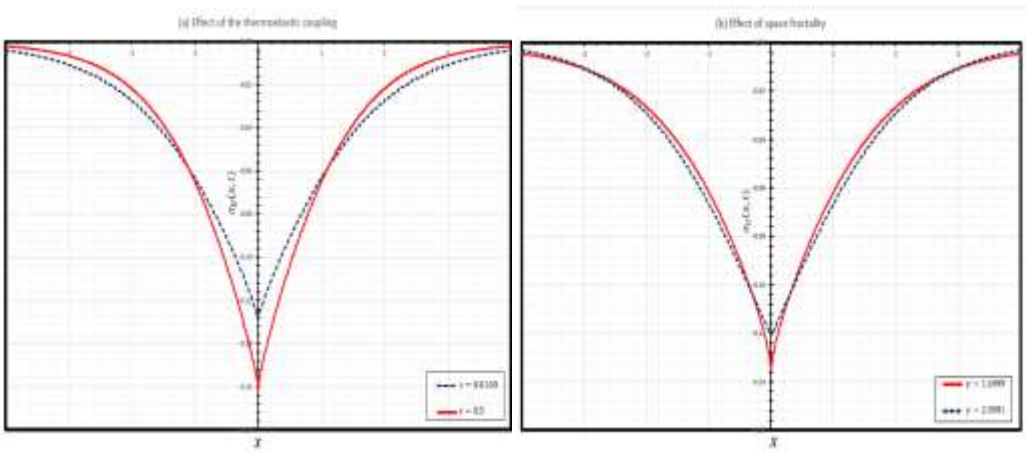

Figure 2: Effect of the thermoelastic coupling ε and the space-fractality γ on the hydrostatic stress at the instant $t = 1$. In (a) Hydrostatic stress at two values of ε and $\gamma = 1.799$; (b) Hydrostatic stress at two values of γ and $\varepsilon = 0.0168$.

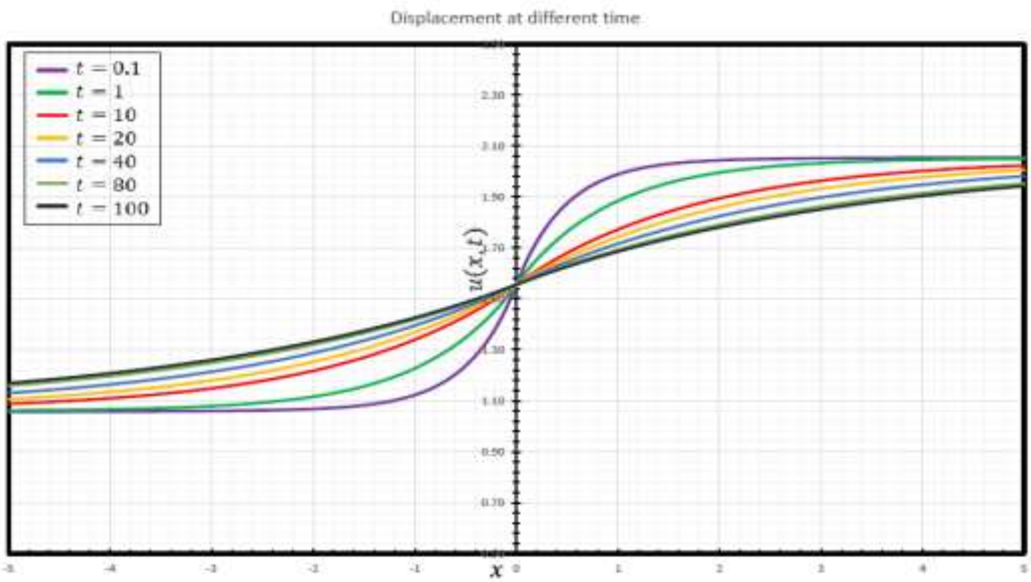

Figure 3: Spatial evolution of displacement at different values of time. The fractional parameters are chosen as $\alpha = 0.70001$ and $\gamma = 1.799$. In the first series $t = 0.1$, the second $t = 1$, the third $t = 10$, the fourth $t = 20$, the fifth $t = 40$, the sixth $t = 80$ and finally $t = 100$.

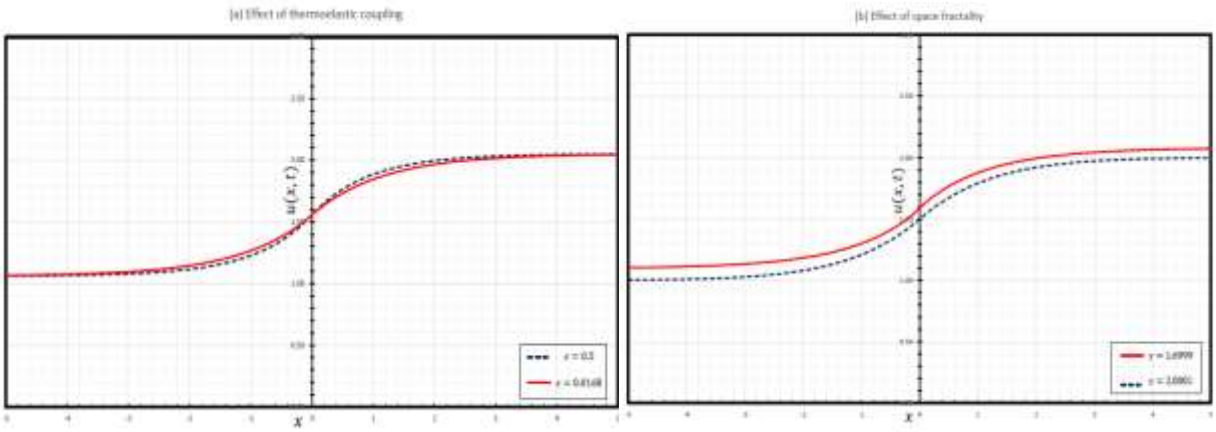

Figure 4: Effect of the thermoelastic coupling ε and the space-fractality γ on the displacement at the instant $t = 1$. In (a) Displacement at two values of ε, and $\gamma = 1.799$; (b) Displacement at two values of γ, and $\varepsilon = 0.0168$.

4. Summary

This paper has examined a space-time fractional Fourier law with a Riemann-Liouville-type time fractional derivative and a single-space fractional derivative of the Riesz type. We have solved an initial value problem with initial conditions on the temperature and the stress σ_{xx}. However, we have noticed that a boundary condition at infinity is mandatory to solve the problem; therefore, the initial value problem is transformed into an initial boundary value problem. The Laplace and Fourier transforms are used to solve the problem, and we have obtained exact solutions in terms of the Fox functions. The effect of anomalous thermal conductivity on the deformation has been studied.

Appendix A. Fractional calculus

This appendix includes a concise summary of the Riemann-Liouville (RL) and Caputo (C) fractional derivatives and integrals and Riesz fractional derivative. For a more in-depth exploration of fractional derivatives, including additional properties, proofs, and other types, readers are encouraged to consult the references [53] and [54].

The left-sided Riemann-Liouville fractional derivative of order α of a well-behaved function $f(t)$ on the interval $[a, b]$ is defined as follows:

$$
{}_{t}^{RL}\mathcal{D}_{a^{+}}^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} \displaystyle\int_a^t \frac{f(\xi)}{(t-\xi)^{\alpha+1-n}}\, d\xi, & n-1<\alpha<n, \\ \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha = n,\ n \in \mathbb{N}, \end{cases} \tag{A.1}
$$

And the left sided Riemann-Liouville fractional integral of order α for a function $f(t)$ defined as follows:

$$
{}_{t}^{RL}I_{a+}^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, & \alpha > 0\ , \\ f(t), & \alpha = 0. \end{cases} \tag{A.2}
$$

The left-sided Caputo fractional derivative acts on functions $f(t)$ on the interval $[a, b]$ and involves a fractional order of differentiation, $\alpha \epsilon (0,1]$ is defined as follows:

$$
{}_{a^{+}}^{C}\mathcal{D}_{t}^{\alpha} f(t) := \begin{cases} {}_{0}^{RL}I_{t}^{1-\alpha}\, \partial_t f(t), & 0<\alpha<1, \\ \dfrac{df(t)}{\mathrm{d}t}, & \alpha = 1. \end{cases} \tag{A.3}
$$

The two fractional derivatives are interconnected through a specific relationship as follows:

$$
{}^{C}\mathcal{D}_{t}^{\alpha}\{f(t)\} = {}^{RL}\mathcal{D}_{t}^{\alpha} f(t) - \frac{f(0^{+})}{\Gamma(1-\alpha)} t^{-\alpha}, \qquad 0<\alpha<1. \tag{A.4}
$$

The Laplace transform for the Riemann–Liouville fractional derivative of a function $f(t)$ is introduced as:

$$
\mathcal{L}\{{}^{RL}\mathcal{D}_{t}^{\alpha} f(t); s\} := s^{\alpha}\tilde{f}(s) - \sum_{i=0}^{k-1} \left[{}^{RL}\mathcal{D}_{t}^{i}\ {}^{RL}I_{t}^{k-\alpha}\right] f(0^{+}) s^{k-1-i}, \quad -1<\alpha\le k. \tag{A.5}
$$

The Laplace transform for Caputo fractional derivative is introduced as:

$$
\mathcal{L}\{{}_{0}^{C}\mathcal{D}_{t}^{\alpha} f(t); s\} := s^{\alpha}\tilde{f}(s) - \sum_{i=0}^{k-1} s^{k-i-1} f^{(i)}(0^{+}), \tag{A.6}
$$

where $k-1<\alpha\le k, \alpha, s \in \mathbb{C}, \mathcal{R}e(s\} > 0, \mathcal{R}e\{\alpha\} > 0$ and $f^{(i)}(0^{+})$ is defined as $\lim_{t\to 0^{+}} f^{(i)}(t)$.

The Riesz fractional derivative operator for the finite interval $[0, L]$ and fractional order $\alpha \epsilon (n-1, n]$, is defined as following:

$$\frac{\partial^{\alpha} f(x,t)}{\partial |x|^{\alpha}} := -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)} \left[{}_{0}^{RL}\mathcal{D}_{x}^{\alpha} + {}_{x}^{RL}\mathcal{D}_{L}^{\alpha} \right] f(x,t), \quad \alpha \neq 1, \tag{A.7}$$

where ${}_{0}^{RL}\mathcal{D}_{x}^{\alpha}$ is the left-sided Riemann-Liouville fractional derivative, and ${}_{x}^{RL}D_{L}^{\alpha}$ is the right-sided Riemann-Liouville fractional derivative.

Appendix B. Special type functions

The probability density function (PDF) satisfies the following two conditions: [55]

1. Non-negativity condition namely, the function must never produce negative values, and can be represented mathematically as:

$$f(x) \geq 0, \tag{B.1}$$

for all x.

2. Normalization which means that the total area under the curve of the function must equal 1, which guarantees that the probabilities sum is equal to 1.
   which can be represented mathematically as:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1. \tag{B.2}$$

The completely monotone function is the function $f(t)$, that takes positive numbers, and its output is either zero or positive. Additionally, the set of completely monotone functions is denoted by CMF. Moreover, a CMF is a smooth, positive-valued function such that increasing the order of differentiation never changes the sign of the output.

The CMF has the following properties:

1. If a function (ϕ) belongs to CMF and there exists another non-negative function (f) such that $\phi(\lambda)$ can be obtained through a specific mathematical operation involving $f(t)$ and the Laplace transform as:

$$\int_{0}^{\infty} e^{-\lambda t} f(t) dt, \quad \lambda > 0. \tag{B.3}$$

2. Multiplying two complete monotone functions will always return a complete monotone function. Furthermore, if we take a linear combination for two complete monotone functions by multiplying each

function into positive constants will also return a complete monotone function.

The set of Stieltjes functions denoted by SF and a function (ϕ) is said to be a Stieltjes function if it can be generated from another function (f) that is a CMF, this construction involves an integral as

$$\int_0^{\infty} e^{-\lambda t} f(t)dt, \;\; \lambda > 0. \tag{B.4}$$

And since $f(t)$ is always non-negative $(f(t) \geq 0)$, the resulting SF (ϕ) will also have a non-negative output for all positive input values. Which makes all Stieltjes functions automatically a subset of Complete monotone function (SF $\subset$ CMF).

The SF has the following properties:

1. If we take a linear combination for two Stieltjes functions by multiplying each function into positive constants will also return a Stieltjes functions.
2. Multiplying two Stieltjes functions doesn't potently guarantee that the resulting function will be a Stieltjes function. But there is a certain condition that allows you to get a SF from multiplication as if we have two SF $\phi(\lambda)$ and $\psi(\lambda)$, then $\phi(\lambda)^{\alpha}\psi(\lambda)^{\beta} \epsilon$ SF, given that $\alpha, \beta \epsilon (0,1) \; and \; \alpha + \beta \leq 1.$

A Bernstein function f satisfies the condition $(-1)^{n-1} f^{(n)}(t) \geq 0$ where n is a natural number, and the set of Bernstein functions is denoted by BF.

The BF has the following properties:

1. A function $\phi(\lambda)$ qualifies as a Bernstein function if and only if, taking its derivative once with respect to λ always results in a completely monotone function.
2. If we take two Bernstein function say $\phi(\lambda)$ and $\psi(\lambda)$ and add them together as $a \; \phi(\lambda) + b \; \psi(\lambda)$, where a and b are any positive numbers, the resulting function will also be a Bernstein function.
3. If we take two Bernstein functions $\phi(\lambda)$ and $\psi(\lambda)$ and compose them together as $(\phi \circ \psi)$, the resulting function will also be a Bernstein function.
4. If we have a completely monotone function ϕ and a Bernstein function ψ, then composing them $(\phi \circ \psi)$ will always result in a function that is completely monotone function.

5. If we have a Bernstein function $\phi(\lambda)$, dividing it by λ always give us a completely monotone function.

A Bernstein function is said to be a complete Bernstein function if and only if, dividing it by λ leads to a Stieltjes function. As if $\psi(\lambda)$ be a BF, then $\psi(\lambda)/\lambda$ is a SF. Noting that the set of complete Bernstein function that is denoted by CBF, is a subset of the set of Bernstein function BF but the opposite is not guaranteed to be true.

The CBF has the following properties:

1. A function is said to be a complete Bernstein function if and only if, taking its reciprocal results as a Stieltjes function. There is one necessary condition: neither the original function nor its reciprocal are identically vanishing functions.
2. If we take two complete Bernstein functions say $\phi(\lambda)$ and $\psi(\lambda)$ and add them together as $a\ \phi(\lambda) + \ b\ \psi(\lambda)$, where a and b are any positive numbers, the resulting function will also be a complete Bernstein function.
3. If we take two complete Bernstein functions say ϕ and ψ, then composing them $(\phi\ \circ\ \psi)$ will always result in a function that is complete Bernstein function**.**
4. If we have a complete Bernstein function $\phi(\lambda)$, then taking its reciprocal $\lambda/\phi(\lambda)$ will also result as a complete Bernstein function. This property applies on the condition that the original function $\phi(\lambda)$ is not identically equal to zero.
5. If we take a complete Bernstein function $\phi(\lambda)$ and compose it with a Stieltjes function $\psi(\lambda)$ or the other way around $\psi(\lambda)\ \circ\ \phi(\lambda)$, The outcome of this combination is also a Stieltjes function.
6. If $\phi(\lambda)$ is a complete Bernstein function then, $\frac{1}{a+b\phi(\lambda)}\ \epsilon$ SF where a and b are positive numbers.
7. The multiplying of two complete Bernstein functions ϕ and ψ is not guaranteed to produce another complete Bernstein function but if we take two complete Bernstein functions ϕ and ψ and apply a small exponent to each function between 0 and 1 say a and b, then multiplying them together as $[\phi(\lambda)]^a\ [\psi(\lambda)]^b$ so, the resulting product remains complete Bernstein function. However, there is a certain condition: the sum of those exponents $(a + b)$ must be less than or equal to 1. [56], [19]

Appendix C. Mittag-Leffler functions

The classical Mittag-Leffler function of one parameter defined as: [50]

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \ , \ z \in \mathbb{C} \ , \mathcal{R}e\{\alpha\} > 0, \tag{C.1}$$

the Mittag-Leffler function of two parameters defined as:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \ , \ z \in \mathbb{C} \ , \mathcal{R}e\{\alpha\}, \mathcal{R}e\{\beta\} > 0. \tag{C.2}$$

Clarifying,

$$E_{\alpha,1}(z) = E_\alpha(z).$$

Additionally, in order to get Laplace transform for Mittag-Leffler we can use the following formulae:

$$\mathcal{L}\{E_\alpha(\lambda t^\alpha); s\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda}, \tag{C.3}$$

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(\lambda x^\alpha); s\right\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \tag{C.4}$$

and

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha); s\right\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \tag{C.5}$$

where,

$$\mathcal{R}e(\beta) > 0, \mathcal{R}e(\alpha) > 0, |\lambda s^{-\alpha}| < 1.$$

The Mellin transforms of the Mittag Leffler functions $E_\alpha(r)$ and $E_{\alpha,\beta}(r)$ defined as follows:

$$\mathcal{M}\{E_\alpha(-r); z\} := \frac{\Gamma(1-z)\Gamma(z)}{\Gamma(1-\alpha z)}, 0 < \mathcal{R}e(z) < 1. \tag{C.6}$$

And

$$\mathcal{M}\{E_{\alpha,\beta}(-r)\,;z\} := \frac{\Gamma(1-z)\Gamma(z)}{\Gamma(\beta-\alpha z)}, 0 < \mathcal{R}e(z) < 1. \tag{C.7}$$

Appendix D. The Fox H-function

The H-function is expressed mathematically by relying on a Mellin-Barnes type integral, as the following: [51]

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p, A_p)\\(b_p, B_p)\end{matrix}\right.\right] \\ &= H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, A_1), \dots, (a_p, A_p)\\(b_1, B_1), \dots, (b_p, B_p)\end{matrix}\right.\right] \\ &= \frac{1}{2\pi i}\int_L \emptyset(s) z^{-s} ds, \end{aligned} \tag{D.1}$$

clarifying that $i = \sqrt{-1}$, z does not equal to zero, $z^{-s} = e^{-s\{ln|z|+iargz\}}$ and

$$\emptyset(s) = \frac{\{\Pi_{j=1}^{m}\Gamma(b_j + B_j s)\}\{\Pi_{j=1}^{n}\Gamma(1 - a_j - A_j s)\}}{\{\Pi_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\}\{\Pi_{j=n+1}^{p}\Gamma(a_j + A_j s)\}}, \tag{D.2}$$

$m, n, p,$ and q are integers satisfying $0 \le n \le p$, $1 \le m \le q$, $a_i, b_j \in \mathbb{C}$, $A_i, B_j \in \mathbb{R}_+$, $i = 1, \cdots, p$, $j = 1, \cdots, q$ such that $A_i(b_j + k) \neq B_j(a_j - l - 1), k, l \in \mathbb{N}; i = 1, \dots, n; j = 1, \dots, m$. The contour L, starts from $\gamma - i\infty$ and ends at $\gamma + i\infty$ and separates the simple poles of $\Gamma(b_j + B_j s), j = 1, \dots, m$ from those of $\Gamma(1 - a_i - A_i s), i = 1, \dots, n$.

The H-function, employed in this study, possesses the following characteristics:

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = \mu H_{p,q}^{m,n}\left[z^{\mu}\left|\begin{matrix}(a_p, \mu A_p)\\(b_q, \mu B_q)\end{matrix}\right.\right], \mu > 0. \tag{D.3}$$

The Euler transform of the Fox H-function is defined as:

$$\begin{aligned} &\int_0^t x^{\rho-1}(t-x)^{\sigma-1} H_{p,q}^{m,n}\left[bx^k\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] \mathrm{d}x \\ &= t^{\rho+\sigma-1}\Gamma(\sigma) H_{p+1,q+1}^{m,n+1}\left[bt^k\left|\begin{matrix}(1-\rho, k), (a_p, A_p)\\(b_q, B_q), (1-\rho-\sigma, k)\end{matrix}\right.\right], \end{aligned} \tag{D.4}$$

where $\rho, \sigma, b \in \mathbb{C}$, and $k > 0$.

The Mellin transform of the H-function is defined as:

$$\int_0^\infty x^{z-1} H_{p,q}^{m,n}\left[\lambda x \left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] dx = \lambda^{-z} \frac{[\Pi_{j=1}^{m}\Gamma(b_j + B_j z)][\Pi_{j=1}^{n}\Gamma(1 - a_j - A_j z)]}{\left[\Pi_{j=m+1}^{q}\Gamma(1 - b_j - B_j z)\right][\Pi_{j=n+1}^{p}\Gamma(a_j + A_j z)]}, \tag{D.5}$$

where $\lambda, s \epsilon \mathbb{C}$; $\min\limits_{1\le j\le m} \mathcal{R}e\left(\frac{b_i}{B_j}\right) < \mathcal{R}e(z) < \max\limits_{1\le j\le n}\left(\frac{1-\mathcal{R}e(a_i)}{A_j}\right), |\arg\lambda| < \frac{1}{2}\pi\alpha, \alpha > 0.$

The inversion of Fourier transformation for H-function $H_{1,2}^{1,1}\left[b|k|^\delta\right]$ is given by the following relation: [19]

$$\mathcal{F}^{-1}\{|q|^\lambda H_{1,2}^{1,1}\left[b|q|^\delta\left|\begin{matrix}(-n,1)\\(0,1);(B,\gamma)\end{matrix}\right.\right];x\} = \frac{1}{\sqrt{4\pi}b^{\frac{\lambda+1}{\delta}}} H_{2,3}^{2,1}\left[\frac{|x|^\delta}{2^\delta b}\left|\begin{matrix}\left(1-\frac{\lambda+1}{\delta},1\right);(1-\beta-\frac{\lambda+1}{\delta}\gamma,\gamma)\\ \left(0,\frac{\delta}{2}\right),\left(1+n-\frac{\lambda+1}{\delta},1\right);(\frac{1}{2},\frac{\delta}{2})\end{matrix}\right.\right], \tag{D.6}$$

where q is the Fourier variable, $b, \gamma, \delta \in \mathbb{R}_+$, $\beta, n \in \mathbb{C}$ and $\lambda \in \mathbb{R}_+ \cup$.

The H-function encompasses the derivative of the Mittag-Leffler function as a particular instance.

$$E_{\alpha,\beta}^{(k)}(z) \coloneqq H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(-k,1)\\(0,1),(1-(\alpha k+\beta),\alpha)\end{matrix}\right.\right], \tag{D.7}$$

or alternatively;

$$E_{\alpha,\beta}^{\gamma}(-z) \coloneqq \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1}\left[z\left|\begin{matrix}(1-\gamma,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right]. \tag{D.8}$$

Refrences

1. J. M. Duhamel, *Second memoire sur les phenomenes thermo-mecaniques*, Journal de l'École polytechnique **15** (1837), no. 25, 1-57.
2. M. A. Biot, *Thermoelasticity and irreversible thermodynamics*, Journal of applied physics **27** (1956), no. 3, 240-253.
3. C. Cattaneo, *Sulla conduzione del calore*, Atti Sem. Mat. Fis. Univ. Modena **3** (1948), 83-101.
4. D. E. Glass, M. N. Özişik and B. Vick, *Hyperbolic heat conduction with surface radiation*, International Journal of Heat and Mass Transfer **28** (1985), no. 10, 1823-1830.
5. A. E. Green and P. M. Naghdi, *On undamped heat waves in an elastic solid*, Journal of Thermal Stresses **15** (1992), no. 2, 253-264.
6. W. Dreyer and H. Struchtrup, *Heat pulse experiments revisited*, Continuum Mechanics and Thermodynamics **5** (1993), no. 1, 3-50.
7. P. Puri and P. K. Kythe, *Nonclassical thermal effects in stokes' second problem*, Acta Mechanica **112** (1995), no. 1, 1-9.
8. J. I. Richard B. Hetnarski, *Generalized thermoelasticity*, Journal of Thermal Stresses **22** (1999), no. 4-5, 451-476.
9. H. W. Lord and Y. Shulman, *A generalized dynamical theory of thermoelasticity*, Journal of the Mechanics and Physics of Solids **15** (1967), no. 5, 299-309.
10. I. Müller, *The coldness, a universal function in thermoelastic bodies*, Archive for Rational Mechanics and Analysis **41** (1971), no. 5, 319-332.
11. A. E. Green and N. Laws, *On the entropy production inequality*, Archive for Rational Mechanics and Analysis **45** (1972), no. 1, 47-53.
12. A. E. Green and K. A. Lindsay, *Thermoelasticity*, Journal of Elasticity **2** (1972), no. 1, 1-7.
13. P. Ponnusamy, *Wave propagation in a generalized thermoelastic solid cylinder of arbitrary cross-section*, International Journal of Solids and Structures **44** (2007), no. 16, 5336-5348.
14. J. Ignaczak, *Generalized thermoelasticity and its applications*, Thermal stresses **3** (1989), 279-354.
15. R. B. Hetnarski and J. Ignaczak, *Soliton-like waves in a low temperature nonlinear thermoelastic solid*, International Journal of Engineering Science **34** (1996), no. 15, 1767-1787.

16. A. E. Green and P. Naghdi, *A re-examination of the basic postulates of thermomechanics*, Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences **432** (1991), no. 1885, 171-194.
17. D. Y. Tzou, *A unified field approach for heat conduction from macro- to micro-scales*, Journal of Heat Transfer **117** (1995), no. 1, 8-16.
18. D. S. Chandrasekharaiah, *Hyperbolic thermoelasticity: A review of recent literature*, Applied Mechanics Reviews **51** (1998), no. 12, 705-729.
19. E. Awad and R. Metzler, *Crossover dynamics from superdiffusion to subdiffusion: Models and solutions*, Fractional Calculus and Applied Analysis **23** (2020), no. 1, 55-102.
20. E. Awad, *Dual-phase-lag in the balance: Sufficiency bounds for the class of jeffreys' equations to furnish physical solutions*, Int J Heat Mass Trans **158** (2020), 119742.
21. E. Awad, T. Sandev, R. Metzler and A. Chechkin, *From continuous-time random walks to the fractional jeffreys equation: Solution and properties*, Int J Heat Mass Transf **181C** (2021), no. December 2021, 121839.
22. —, *Closed-form multi-dimensional solutions and asymptotic behaviors for subdiffusive processes with crossovers: I. Retarding case*, Chaos, Solitons & Fractals **152C** (2021), 111357.
23. E. Awad and R. Metzler, *Closed-form multi-dimensional solutions and asymptotic behaviours for subdiffusive processes with crossovers: Ii. Accelerating case*, Journal of Physics A: Mathematical and Theoretical **55** (2022), no. 20, 205003.
24. E. Awad, *Modeling of anomalous thermal conduction in thermoelectric magnetohydrodynamics: Couette formulation with a multiphase pressure gradient*, Phys. Fluids **36** (2024), no. 3.
25. E. Bazhlekova and I. Bazhlekov, *Transition from diffusion to wave propagation in fractional jeffreys-type heat conduction equation*, Fractal Fractional **4** (2020), no. 3, 32.
26. E. Bazhlekova and I. Bazhlekov, *Identification of a space-dependent source term in a nonlocal problem for the general time-fractional diffusion equation*, Journal of Computational applied Mathematics **386** (2021), 113213.
27. E. Bazhlekova, *Completely monotone multinomial mittag-leffler type functions and diffusion equations with multiple time-derivatives*, Fractional Calculus Applied Analysis **24** (2021), no. 1, 88-111.

28. K. Górska and A. Horzela, *Subordination and memory dependent kinetics in diffusion and relaxation phenomena*, Fractional Calculus Applied Analysis (2023), 1-33.

29. T. Pietrzak, A. Horzela and K. Górska, *The generalized telegraph equation with moving harmonic source: Solvability using the integral decomposition technique and wave aspects*, International Journal of Heat Mass Transfer **225** (2024), 125373.

30. Y. Z. Povstenko, *Fractional heat conduction equation and associated thermal stress*, Journal of Thermal Stresses **28** (2005), no. 1, 83-102.

31. H. H. Sherief, A. M. A. El-Sayed and A. M. Abd El-Latief, *Fractional order theory of thermoelasticity*, International Journal of Solids and Structures **47** (2010), no. 2, 269-275.

32. H. M. Youssef, *Theory of fractional order generalized thermoelasticity*, Journal of Heat Transfer **132** (2010), no. 6, 1-7.

33. Y. Z. Povstenko, *Fractional cattaneo-type equations and generalized thermoelasticity*, Journal of Thermal Stresses **34** (2011), no. 2, 97-114.

34. M. A. Ezzat and A. S. E. Karamany, *Fractional order heat conduction law in magneto-thermoelasticity involving two temperatures*, Zeitschrift fur Angewandte Mathematik und Physik **62** (2011), no. 5, 937-952.

35. E. Awad, *On the generalized thermal lagging behavior: Refined aspects*, Journal of Thermal Stresses **35** (2012), no. 4, 293-325.

36. Y. Z. Povstenko, *Theory of thermoelasticity based on the space-time-fractional heat conduction equation*, 2009, p.^pp.

37. H. H. Sherief and E. M. Hussein, *Fractional order model of micropolar thermoelasticity and 2d half-space problem*, Acta Mechanica **234** (2023), no. 2, 535-552.

38. E. Awad, W. Dai and S. Sobolev, *Thermal oscillations and resonance in electron–phonon interaction process*, Zeitschrift für angewandte Mathematik und Physik **75** (2024), no. 4, 143.

39. D. E. Carlson, "Linear thermoelasticity," *Linear theories of elasticity and thermoelasticity: Linear and nonlinear theories of rods, plates, and shells*, C. Truesdell (Editor), Springer Berlin Heidelberg, Berlin, Heidelberg, 1973, pp. 297-345.

40. J. z. Ignaczak and M. Ostoja-Starzewski, "Thermoelasticity with finite wave speeds," Oxford University Press, Oxford, 2010.

41. Y. Povstenko, "Fractional heat conduction and related theories of thermoelasticity," *Fractional thermoelasticity*, Y. Povstenko (Editor), Springer International Publishing, Cham, 2015, pp. 13-33.

42. A. I. Saichev and G. M. Zaslavsky, *Fractional kinetic equations: Solutions and applications*, Chaos: An Interdisciplinary Journal of Nonlinear Science **7** (1997), no. 4, 753-764.

43. F. Mainardi, Y. Luchko and G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, arXiv preprint cond-mat/0702419 (2007).

44. R. Gorenflo and F. Mainardi, "Fractional calculus," *Fractals and fractional calculus in continuum mechanics*, A. Carpinteri and F. Mainardi (Editors), Springer Vienna, Vienna, 1997, pp. 223-276.

45. T. Sandev and Z. Tomovski, *Fractional equations and models*, Theory and applications. Cham, Switzerland: Springer Nature Switzerland AG (2019).

46. Q. Yang, F. Liu and I. Turner, *Numerical methods for fractional partial differential equations with riesz space fractional derivatives*, Applied Mathematical Modelling **34** (2010), no. 1, 200-218.

47. A. Compte and R. Metzler, *The generalized cattaneo equation for the description of anomalous transport processes*, Journal of Physics A: Mathematical and General **30** (1997), no. 21, 7277.

48. W. Zhang and Y. Zhou, "Chapter 2 - level-set functions and parametric functions," *The feature-driven method for structural optimization*, W. Zhang and Y. Zhou (Editors), Elsevier, 2021, pp. 9-46.

49. F. Mainardi, A. Mura and G. Pagnini, *The m-wright function in time-fractional diffusion processes: A tutorial survey*, International Journal of Differential Equations **2010** (2010), no. 1, 104505.

50. R. Garra and R. Garrappa, *The prabhakar or three parameter mittag–leffler function: Theory and application*, Communications in Nonlinear Science and Numerical Simulation **56** (2018), 314-329.

51. A. M. Mathai, R. K. Saxena and H. J. Haubold, *The h-function: Theory and applications*, Springer Science & Business Media, 2009.

52. E. Awad and N. Samir, *A closed-form solution for thermally induced affine deformation in unbounded domains with a temporally accelerated anomalous thermal conductivity*, Journal of Physics A: Mathematical and Theoretical **57** (2024), no. 45, 455202.

53. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, elsevier, 1998.
54. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204, elsevier, 2006.
55. P. Olofsson and M. Andersson, *Probability, statistics, and stochastic processes*, John Wiley & Sons, 2012.
56. E. Awad, *On the time-fractional cattaneo equation of distributed order*, Physica A: Statistical Mechanics and its Applications **518** (2019), 210-233.